

Comment on ‘The $\tan \theta$ theorem with relaxed conditions’, by Y. Nakatsukasa

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ABSTRACT. We show that in case of the spectral norm, one of the main results of the recent paper *The $\tan \theta$ theorem with relaxed conditions*, by Yuji Nakatsukasa, published in *Linear Algebra and its Applications* is a corollary of the $\tan \theta$ theorem proven in [V. Kostykin, K. A. Makarov, and A. K. Motovilov, *On the existence of solutions to the operator Riccati equation and the $\tan \theta$ theorem*, IEOT **51** (2005), 121 – 140]. We also give an alternative finite-dimensional matrix formulation of another $\tan \theta$ theorem proven in [S. Albeverio and A. K. Motovilov, *The a priori $\tan \theta$ theorem for spectral subspaces*, IEOT **73** (2012), 413 - 430].

In a recent paper [7] published in *Linear Algebra and its Applications*, Y. Nakatsukasa obtains two bounds on the tangent of the canonical angles between an approximate and an exact spectral subspace of a Hermitian matrix. These bounds (see [7, Theorems 1 and 2]) extend respectively the $\tan \theta$ theorem and the generalized $\tan \theta$ theorem proven by C. Davis and W. M. Kahan in their celebrated paper [2]. Actually, an extension of the $\tan \theta$ theorem similar to [7, Theorem 1] has already been given in [4], in the wider context of the perturbation theory for self-adjoint operators on a Hilbert space.

In our discussion below we restrict ourselves to the spectral norms of the matrices involved, that is, by $\|S\|$ we always understand the maximal singular value of a matrix S . If \mathfrak{A} and \mathfrak{L} are subspaces of \mathbb{C}^n , the notation $\angle(\mathfrak{A}, \mathfrak{L})$ is used for the largest principal angle between \mathfrak{A} and \mathfrak{L} .

We begin with presenting a relevant finite-dimensional version of the $\tan \theta$ theorem from [4] (see [4, Theorem 2]).

Proposition 1. Assume that a Hermitian matrix $L \in \mathbb{C}^{n \times n}$ is block partitioned in the form

$$L = \begin{bmatrix} A_1 & B^H \\ B & A_2 \end{bmatrix} \quad (1)$$

with $A_1 \in \mathbb{C}^{k \times k}$, $1 < k < n$. Let the spectrum of A_1 lie in $(-\infty, \alpha - \delta] \cup [\beta + \delta, \infty)$, where $\alpha \leq \beta$ and $\delta > 0$. Suppose that \mathfrak{L}_1 and \mathfrak{L}_2 are complementary orthogonal reducing subspaces of L such that $\dim(\mathfrak{L}_1) = k$ and the spectrum of the restriction $L|_{\mathfrak{L}_2}$ of (the operator) L on the reducing subspace \mathfrak{L}_2 is confined in $[\alpha, \beta]$. Also, let \mathfrak{A}_1 be the subspace of \mathbb{C}^n spanned by the first k columns of the identity matrix I_n . Then

$$\tan \angle(\mathfrak{A}_1, \mathfrak{L}_1) \leq \frac{\|B\|}{\delta}. \quad (2)$$

Remark 2. Actually, [4, Theorem 2] (combined with [4, Theorem 2.3]) suggests the equivalent bound $\delta \tan \|\Theta\| \leq \|B\|$ for the operator angle Θ between the orthogonal complements \mathfrak{A}_2 and \mathfrak{L}_2 of the subspaces \mathfrak{A}_1 and \mathfrak{L}_1 , respectively, provided that \mathfrak{L}_2 is the graph of an operator from \mathfrak{A}_2 to \mathfrak{A}_1 . But the latter, in the finite-dimensional case under consideration, holds true automatically. This is seen from the following lemma.

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Lemma 3. *Assume the hypothesis of Proposition 1. Then $\mathfrak{A}_2 \cap \mathfrak{L}_1 = \mathfrak{A}_1 \cap \mathfrak{L}_2 = \{0\}$ and, hence, the reducing subspace \mathfrak{L}_2 is the graph of an operator from \mathfrak{A}_2 to \mathfrak{A}_1 .*

Proof. By the hypothesis, the dimensions of the subspaces \mathfrak{L}_1 and \mathfrak{A}_1 coincide, $\dim(\mathfrak{L}_1) = \dim(\mathfrak{A}_1) = k$. Then by using the canonical orthogonal decomposition of \mathbb{C}^n with respect to the orthogonal projections onto \mathfrak{A}_1 and \mathfrak{L}_1 (see, e.g. [3, Theorem 2.2]) one verifies that

$$\dim(\mathfrak{A}_2 \cap \mathfrak{L}_1) = \dim(\mathfrak{A}_1 \cap \mathfrak{L}_2). \quad (3)$$

Suppose that $\mathfrak{A}_1 \cap \mathfrak{L}_2 \neq \{0\}$. In such a case, there is a vector $y \in \mathfrak{L}_2$ of the form $y = \begin{bmatrix} x \\ 0_{n-k} \end{bmatrix}$, where the lower subcolumn 0_{n-k} consists of exactly $n - k$ zeros and the upper subcolumn x contains at least one nonzero element. For $c = (\alpha + \beta)/2$ one then obtains

$$\|(L - cI_n)y\|^2 = \|(A_1 - cI_k)x\|^2 + \|Bx\|^2 \geq \|(A_1 - cI_k)x\|^2 \geq \left(\frac{1}{2}(\beta - \alpha) + \delta\right)^2 \|y\|^2,$$

since $\|y\| = \|x\|$ and the spectrum of A_1 belongs to $(-\infty, \alpha - \delta] \cup [\beta + \delta, \infty)$. On the other hand, for $y \in \mathfrak{L}_2$ we should have $\|(L - cI_n)y\| \leq \frac{1}{2}(\beta - \alpha)\|y\|$ since the spectrum of the restriction $L|_{\mathfrak{L}_2}$ lies in $[\alpha, \beta]$. Hence, $y = 0$, a contradiction, which yields $\mathfrak{A}_1 \cap \mathfrak{L}_2 = \{0\}$. Taking into account (3) one concludes that also $\mathfrak{A}_2 \cap \mathfrak{L}_1 = \{0\}$. Applying [3, Theorem 3.2] completes the proof. \square

Now we show that for the spectral norm the $\tan \theta$ theorem proven in [7] is a corollary of Proposition 1. We reproduce the corresponding statement from [7] in the following form (see [7, Theorem 1]).

Proposition 4 ([7]). *Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix. Let $X = [X_1 \ X_2]$ be a unitary eigenvector matrix of A with $X_1 \in \mathbb{C}^{n \times k}$, $1 < k < n$, so that $X^H A X = \text{diag}(\Lambda_1, \Lambda_2)$ is diagonal and Λ_1 has k columns. Assume that the columns of a matrix $Q_1 \in \mathbb{C}^{n \times k}$ are orthonormal and let $R = A Q_1 - Q_1 A_1$, where $A_1 = Q_1^H A Q_1$. Furthermore, assume that for some $\alpha \leq \beta$ and $\delta > 0$ the spectrum of A_1 lies in $(-\infty, \alpha - \delta] \cup [\beta + \delta, \infty)$ and the spectrum of Λ_2 belongs to $[\alpha, \beta]$. Then*

$$\tan \angle(\mathfrak{Q}_1, \mathfrak{X}_1) \leq \frac{\|R\|}{\delta}, \quad (4)$$

where \mathfrak{Q}_1 and \mathfrak{X}_1 are the subspaces spanned by the columns of Q_1 and X_1 , respectively.

Proof. Assume that Q_1 is a submatrix of a unitary $n \times n$ matrix $Q = [Q_1 \ Q_2]$ and let $L = Q^H A Q$. The matrix L has the form (1) with $A_1 = Q_1^H A Q_1$, $A_2 = Q_2^H A Q_2$, and $B = Q_2^H A Q_1$. Since A is unitarily equivalent to the diagonal matrix $\Lambda = \text{diag}(\Lambda_1, \Lambda_2)$, the same is true for L . Moreover, the k -dimensional subspace $\mathfrak{L}_1 = Q^H \mathfrak{X}_1$ and its orthogonal complement $\mathfrak{L}_2 = \mathbb{C}^n \ominus \mathfrak{L}_1$ are reducing subspaces of L . The spectrum of the restriction $L|_{\mathfrak{L}_2}$ coincides with the spectrum of Λ_2 and, hence, it lies in $[\alpha, \beta]$. If the subspace \mathfrak{A}_1 is as in Proposition 1, then, just by this proposition, the largest principal angle between \mathfrak{A}_1 and \mathfrak{L}_1 satisfies the bound (2). Meanwhile, the subspaces \mathfrak{Q}_1 and \mathfrak{X}_1 are obtained from \mathfrak{A}_1 and \mathfrak{L}_1 by the same unitary transformation: $\mathfrak{Q}_1 = Q \mathfrak{A}_1$ and $\mathfrak{X}_1 = Q \mathfrak{L}_1$. Hence, $\angle(\mathfrak{Q}_1, \mathfrak{X}_1) = \angle(\mathfrak{A}_1, \mathfrak{L}_1)$. Observing that $B = Q_2^H (A Q_1 - Q_1 A_1) = Q_2^H R$, one infers $\|B\| = \|R\|$ and then (2) implies (4). \square

Remark 5. In its turn, Proposition 1 may be viewed as a particular version of Proposition 4 for the case where $[Q_1 \ Q_2]$ is taken equal to the identity matrix I_n . Thus, in fact these two propositions are equivalent to each other.

We next note that there is another sharp $\tan \theta$ bound established in [1, Theorem 1] (see also [6, Theorem 2] for an earlier result). The following assertion represents a finite-dimensional version of [1, Theorem 1] reformulated in the style of Proposition 4.

Proposition 6. *Let $A \in \mathbb{C}^{n \times n}$ be a Hermitian matrix and $Q = [Q_1 \ Q_2]$ a unitary matrix with $Q_1 \in \mathbb{C}^{n \times k}$, $1 < k < n$. Assume that for some $a \leq b$ and $d > 0$ the spectrum of $A_1 = Q_1^H A Q_1$ lies in $(-\infty, a - d] \cup [b + d, \infty)$ and that the spectrum of $A_2 = Q_2^H A Q_2$ belongs to $[a, b]$. Let $R = A Q_1 - Q_1 A_1$ and suppose that $\|R\| < \sqrt{2}d$. Then n orthonormal eigenvectors of A may be numbered in such an order that the corresponding unitary eigenvector matrix $X = [X_1 \ X_2]$ with $X_1 \in \mathbb{C}^{n \times k}$ reduces A to the diagonal form $X^H A X = \text{diag}(\Lambda_1, \Lambda_2)$ with $\Lambda_1 \in \mathbb{C}^{k \times k}$ having its spectrum in $(-\infty, a - d] \cup [b + d, \infty)$, and with Λ_2 having all its eigenvalues in $[a - \delta_R, b + \delta_R]$, where $\delta_R = \|R\| \tan\left(\frac{1}{2} \arctan \frac{2\|R\|}{d}\right) < d$. Moreover,*

$$\tan \angle(\mathfrak{Q}_1, \mathfrak{X}_1) \leq \frac{\|R\|}{d}, \quad (5)$$

where \mathfrak{Q}_1 and \mathfrak{X}_1 are the subspaces spanned by the columns of Q_1 and X_1 , respectively.

Proof. The matrix $L = Q^H A Q$ has the form (1) with A_1 and A_2 defined in the hypothesis, and $B = Q_2^H A Q_1$. As in the proof of Proposition 4 we have $\|B\| = \|R\|$. Hence $\|B\| < \sqrt{2}d$ and then the statement on the eigenvalue matrix Λ and, in particular, on the spectral inclusions for Λ_1 and Λ_2 , is an immediate corollary of [5, Theorem 2]. Furthermore, for the case under consideration, the bound from [1, estimate (1.3) in Theorem 1] may be equivalently written as $d \tan \angle(\mathfrak{Q}_1, \mathfrak{L}_1) \leq \|R\|$, where \mathfrak{Q}_1 is as in Proposition 1 and \mathfrak{L}_1 is the spectral subspace of L associated with the set $(-\infty, a - d] \cup [b + d, \infty)$. By the unitarity argument we already used in the proof of Proposition 4, the bound $d \tan \angle(\mathfrak{Q}_1, \mathfrak{L}_1) \leq \|R\|$ implies the bound (5). \square

Remark 7. In general, condition $\|R\| < \sqrt{2}d$ cannot be removed. If this condition is violated, the matrix A may not have eigenvalues in the interval $(a - d, b + d)$ at all (see [5, Example 1.6]).

If we estimate $\angle(\mathfrak{Q}_1, \mathfrak{X}_1)$ by using inequality (5), no knowledge on the exact eigenvalues of A is required. Unlike the bound (4), the estimate (5) involves the separation distance d between the respective eigenvalue sets of the matrices A_1 and A_2 . In applications, these sets are usually treated as an approximate spectrum of A and their separation distance is assumed to be known prior to further calculations. Following [6] and [1], it is appropriate thus to call the bound (5) the *a priori* tan θ theorem. Similarly, the bound (4) may be called the *(semi-)a posteriori* tan θ theorem since it involves the separation distance δ between one approximate and one exact spectral sets.

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